

# On the asymptotic expansions of products related to the Wallis, Weierstrass, and Wilf formulas

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**Abstract:** For all integers  $n \geq 1$ , let

$$W_n(p, q) = \prod_{j=1}^n \left\{ e^{-p/j} \left( 1 + \frac{p}{j} + \frac{q}{j^2} \right) \right\}$$

and

$$R_n(p, q) = \prod_{j=1}^n \left\{ e^{-p/(2j-1)} \left( 1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2} \right) \right\},$$

where  $p, q$  are complex parameters. The infinite product  $W_\infty(p, q)$  includes the Wallis and Wilf formulas, and also the infinite product definition of Weierstrass for the gamma function, as special cases. In this paper, we present asymptotic expansions of  $W_n(p, q)$  and  $R_n(p, q)$  as  $n \rightarrow \infty$ . In addition, we also establish asymptotic expansions for the Wallis sequence.

**2010 Mathematics Subject Classification:** 33B15, 41A60

**Keywords:** Gamma function, Psi function, Euler–Mascheroni constant, Asymptotic expansion, Wallis sequence

## 1 Introduction

The famous Wallis sequence  $W_n$ , defined by

$$W_n = \prod_{k=1}^n \frac{4k^2}{4k^2 - 1} \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}), \quad (1.1)$$

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has the limiting value

$$W_\infty = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} = \frac{\pi}{2} \quad (1.2)$$

established by Wallis in 1655; see [5, p. 68]. Several elementary proofs of this well-known result can be found in [3, 23, 37]. An interesting geometric construction that produces the above limiting value can be found in [30]. Many formulas exist for the representation of  $\pi$ , and a collection of these formulas is listed [33, 34]. For more history of  $\pi$  see [2, 4, 5, 14].

The following infinite product definition for the gamma function is due to Weierstrass (see, for example, [1, p. 255, Entry (6.1.3)]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left\{ e^{-z/n} \left( 1 + \frac{z}{n} \right) \right\}, \quad (1.3)$$

where  $\gamma$  denotes the Euler–Mascheroni constant defined by

$$\gamma := \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln n \right) = 0.5772156649 \dots$$

In 1997, Wilf [39] posed the following elegant infinite product formula as a problem:

$$\prod_{j=1}^{\infty} \left\{ e^{-1/j} \left( 1 + \frac{1}{j} + \frac{1}{2j^2} \right) \right\} = \frac{e^{\pi/2} + e^{-\pi/2}}{\pi e^{\gamma}}, \quad (1.4)$$

which contains three of the most important mathematical constants, namely  $\pi$ ,  $e$  and  $\gamma$ . Subsequently, Choi and Seo [12] proved (1.4), together with three other similar product formulas, by making use of well-known infinite product formulas for the circular and hyperbolic functions and the familiar Stirling formula for the factorial function.

In 2003, Choi *et al.* [11] presented the following two general infinite product formulas, which include Wilf's formula (1.4) and other similar formulas in [12] as special cases:

$$\prod_{j=1}^{\infty} \left\{ e^{-1/j} \left( 1 + \frac{1}{j} + \frac{\alpha^2 + 1/4}{j^2} \right) \right\} = \frac{2(e^{\pi\alpha} + e^{-\pi\alpha})}{(4\alpha^2 + 1)\pi e^{\gamma}} \quad \left( \alpha \in \mathbb{C}; \alpha \neq \pm \frac{1}{2}i \right) \quad (1.5)$$

and

$$\prod_{j=1}^{\infty} \left\{ e^{-2/j} \left( 1 + \frac{2}{j} + \frac{\beta^2 + 1}{j^2} \right) \right\} = \frac{e^{\pi\beta} - e^{-\pi\beta}}{2\beta(\beta^2 + 1)\pi e^{2\gamma}} \quad (\beta \in \mathbb{C} \setminus \{0\}; \beta \neq \pm i), \quad (1.6)$$

where  $i = \sqrt{-1}$  and  $\mathbb{C}$  denotes the set of complex numbers. In 2013, Chen and Choi [7] presented a more general infinite product formula that included the formulas (1.5) and (1.6) as special cases:

$$\prod_{j=1}^{\infty} \left\{ e^{-p/j} \left( 1 + \frac{p}{j} + \frac{q}{j^2} \right) \right\} = \frac{e^{-p\gamma}}{\Gamma(1 + \frac{1}{2}p + \frac{1}{2}\Delta) \Gamma(1 + \frac{1}{2}p - \frac{1}{2}\Delta)} \quad (1.7)$$

and also another interesting infinite product formula:

$$\prod_{j=1}^{\infty} \left\{ e^{-p/(2j-1)} \left( 1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2} \right) \right\} = \frac{2^{-p} \pi e^{-p\gamma/2}}{\Gamma\left(\frac{1}{2} + \frac{1}{4}p + \frac{1}{4}\Delta\right) \Gamma\left(\frac{1}{2} + \frac{1}{4}p - \frac{1}{4}\Delta\right)}, \quad (1.8)$$

where  $p, q \in \mathbb{C}$  and  $\Delta := \sqrt{p^2 - 4q}$ .

The formula (1.7) can be seen to include the formulas (1.2)–(1.6) as special cases. By setting  $(p, q) = (0, -1/4)$  in (1.7), we have

$$\prod_{j=1}^{\infty} \left( 1 - \frac{1}{4j^2} \right) = \frac{2}{\pi}, \quad (1.9)$$

whose reciprocal becomes the Wallis product (1.2). Also setting  $q = 0$  in (1.7), we obtain

$$\prod_{j=1}^{\infty} \left\{ e^{-p/j} \left( 1 + \frac{p}{j} \right) \right\} = \frac{e^{-p\gamma}}{\Gamma(p+1)}. \quad (1.10)$$

Noting that  $\Gamma(z+1) = z\Gamma(z)$  and replacing  $p$  by  $z$  in (1.10) we recover the Weierstrass formula (1.3). Setting  $(p, q) = (1, 1/2)$  in (1.7) yields the Wilf formula (1.4) and setting

$$(p, q) = \left( 1, \alpha^2 + \frac{1}{4} \right) \quad \text{and} \quad (p, q) = (2, \beta^2 + 1) \quad (1.11)$$

in (1.7) yields the formulas (1.5) and (1.6), respectively.

With  $(p, q) = (-1, 1/4)$  in (1.7), we obtain the beautiful infinite product formula expressed in terms of the most important constants  $\pi$ ,  $e$  and  $\gamma$ , namely

$$\prod_{j=1}^{\infty} \left\{ e^{1/j} \left( 1 - \frac{1}{2j} \right)^2 \right\} = \frac{e^{\gamma}}{\pi}. \quad (1.12)$$

Also worthy of note are the infinite products that result from setting  $(p, q) = (-2, 0)$  and  $(p, q) = (2, 0)$  in (1.8) to yield respectively

$$\prod_{j=1}^{\infty} \left\{ e^{2/(2j-1)} \left( 1 - \frac{2}{2j-1} \right) \right\} = -2e^{\gamma} \quad (1.13)$$

and

$$\prod_{j=1}^{\infty} \left\{ e^{-2/(2j-1)} \left( 1 + \frac{2}{2j-1} \right) \right\} = \frac{1}{2e^{\gamma}}. \quad (1.14)$$

**Remark 1.1.** The constant  $e^{\gamma}$  is important in number theory and equals the following limit, where  $p_n$  is the  $n$ th prime number:

$$e^{\gamma} = \lim_{n \rightarrow \infty} \frac{1}{\ln p_n} \prod_{j=1}^n \frac{p_j}{p_j - 1}.$$

This restates the third of Mertens' theorems (see [38]). The numerical value of  $e^{\gamma}$  is:

$$e^{\gamma} = 1.7810724179 \dots$$

There is the curious radical representation

$$e^\gamma = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \cdots, \quad (1.15)$$

where the  $n$ th factor is

$$\left( \prod_{k=0}^n (k+1)^{(-1)^{k+1} \binom{n}{k}} \right)^{1/(n+1)}.$$

The product (1.15), first discovered in 1926 by Ser [32], was rediscovered in [16, 35, 36].

Recently, Chen and Paris [9] generalized the formula (1.7) to include  $m$  parameters  $(p_1, \dots, p_m)$ . Subsequently, Chen and Paris [10] considered the asymptotic expansion of products related to generalization of the Wilf problem. However, these authors did not give a general formula for the coefficients in their expansions.

For  $n \in \mathbb{N}$ , let

$$W_n(p, q) = \prod_{j=1}^n \left\{ e^{-p/j} \left( 1 + \frac{p}{j} + \frac{q}{j^2} \right) \right\} \quad (1.16)$$

and

$$R_n(p, q) = \prod_{j=1}^n \left\{ e^{-p/(2j-1)} \left( 1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2} \right) \right\}, \quad (1.17)$$

where  $p$  and  $q$  are complex parameters. In this paper, we present asymptotic expansions of  $W_n(p, q)$  and  $R_n(p, q)$  as  $n \rightarrow \infty$ , including recurrence relations for the coefficients in these expansions. Furthermore, we establish asymptotic expansions for the Wallis sequence  $W_n$ .

## 2 Asymptotic expansions of $W_n(p, q)$ and $R_n(r, s)$

It was established by Chen and Choi [7] that the finite products  $W_n(p, q)$  and  $R_n(p, q)$  defined in (1.16) and (1.17) can be expressed in the following closed form

$$W_n(p, q) = \frac{e^{-p(\psi(n+1)+\gamma)} \Gamma\left(n+1+\frac{1}{2}p+\frac{1}{2}\Delta\right) \Gamma\left(n+1+\frac{1}{2}p-\frac{1}{2}\Delta\right)}{(\Gamma(n+1))^2 \Gamma\left(1+\frac{1}{2}p+\frac{1}{2}\Delta\right) \Gamma\left(1+\frac{1}{2}p-\frac{1}{2}\Delta\right)} \quad (2.1)$$

and

$$R_n(p, q) = \frac{e^{-\frac{p}{2}(\psi(n+\frac{1}{2})+\gamma+2\ln 2)} \Gamma\left(n+\frac{1}{2}+\frac{1}{4}p+\frac{1}{4}\Delta\right) \Gamma\left(n+\frac{1}{2}+\frac{1}{4}p-\frac{1}{4}\Delta\right) \pi}{(\Gamma(n+\frac{1}{2}))^2 \Gamma\left(\frac{1}{2}+\frac{1}{4}p+\frac{1}{4}\Delta\right) \Gamma\left(\frac{1}{2}+\frac{1}{4}p-\frac{1}{4}\Delta\right)}, \quad (2.2)$$

where  $\psi(z)$  denotes the psi (or digamma) function, defined by

$$\psi(z) = \frac{d}{dz} \{\ln \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We observe that allowing  $n \rightarrow \infty$  in (2.1) and (2.2), respectively, yields (1.7) and (1.8).

Define the function  $f(z)$  by

$$f(z) := \frac{e^{-\lambda\psi(z)}\Gamma(z+\mu)\Gamma(z+\nu)}{(\Gamma(z))^2}, \quad (2.3)$$

where  $\lambda, \mu, \nu \in \mathbb{C}$ . It is well known that the logarithm of the gamma function has the asymptotic expansion (see [22, p. 32]):

$$\ln \Gamma(z+a) \sim \left(z+a-\frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}(a)}{n(n+1)} \frac{1}{z^n} \quad (2.4)$$

for  $z \rightarrow \infty$  in  $|\arg z| < \pi$ , where  $B_n(t)$  denote the Bernoulli polynomials defined by the following generating function:

$$\frac{ze^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!}.$$

Note that the Bernoulli numbers  $B_n$  ( $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ) are defined by  $B_n := B_n(0)$ . The psi function has the asymptotic expansion (see [22, p. 33]):

$$\psi(z) \sim \ln z - \frac{1}{z} - \sum_{j=1}^{\infty} \frac{B_j}{jz^j} \quad (z \rightarrow \infty; |\arg z| < \pi). \quad (2.5)$$

Using (2.4) and (2.5), we then find that

$$\ln f(z) \sim (\mu + \nu - \lambda) \ln z + \sum_{j=1}^{\infty} \frac{a_j}{z^j}$$

or

$$f(z) \sim z^{\mu+\nu-\lambda} \exp \left( \sum_{j=1}^{\infty} \frac{a_j}{z^j} \right) \quad (2.6)$$

for  $z \rightarrow \infty$  in  $|\arg z| < \pi$ , where the coefficients  $a_j \equiv a_j(\lambda, \mu, \nu)$  are given by

$$a_1 = \frac{\lambda + B_2(\mu) + B_2(\nu) - 2B_2}{2}, \quad a_j = \frac{\lambda B_j}{j} + \frac{(-1)^{j+1} (B_{j+1}(\mu) + B_{j+1}(\nu) - 2B_{j+1})}{j(j+1)} \quad (j \geq 2). \quad (2.7)$$

The choice

$$(\lambda, \mu, \nu) = \left( p, \frac{1}{2}p + \frac{1}{2}\Delta, \frac{1}{2}p - \frac{1}{2}\Delta \right),$$

where  $\mu + \nu - \lambda = 0$ , leads to the first few coefficients  $a_j(p, q)$  given by:

$$\begin{aligned} a_1(p, q) &= \frac{1}{2}p^2 - q, \\ a_2(p, q) &= -\frac{1}{6}p^3 + \frac{1}{2}pq + \frac{1}{4}p^2 - \frac{1}{2}q, \\ a_3(p, q) &= \frac{1}{12}p^4 - \frac{1}{3}p^2q + \frac{1}{6}q^2 - \frac{1}{6}p^3 + \frac{1}{2}pq + \frac{1}{12}p^2 - \frac{1}{6}q. \end{aligned}$$

From (2.1) and (2.6), we obtain the following

**Theorem 2.1.** As  $n \rightarrow \infty$ , we have

$$W_n(p, q) \sim \frac{e^{-p\gamma}}{\Gamma\left(1 + \frac{1}{2}p + \frac{1}{2}\Delta\right) \Gamma\left(1 + \frac{1}{2}p - \frac{1}{2}\Delta\right)} \exp\left(\sum_{j=1}^{\infty} \frac{a_j(p, q)}{(n+1)^j}\right), \quad (2.8)$$

where the coefficients  $a_j(p, q)$  are given by

$$\begin{aligned} a_1(p, q) &= \frac{1}{2}p^2 - q \quad \text{and} \\ a_j(p, q) &= \frac{pB_j}{j} + \frac{(-1)^{j+1} \left( B_{j+1}(\frac{1}{2}p + \frac{1}{2}\Delta) + B_{j+1}(\frac{1}{2}p - \frac{1}{2}\Delta) - 2B_{j+1} \right)}{j(j+1)} \quad (j \geq 2). \end{aligned} \quad (2.9)$$

Thus we have the expansion

$$\begin{aligned} W_n(p, q) &\sim \frac{e^{-p\gamma}}{\Gamma\left(1 + \frac{1}{2}p + \frac{1}{2}\Delta\right) \Gamma\left(1 + \frac{1}{2}p - \frac{1}{2}\Delta\right)} \\ &\times \exp\left(\frac{\frac{1}{2}p^2 - q}{n+1} + \frac{-\frac{1}{6}p^3 + \frac{1}{2}pq + \frac{1}{4}p^2 - \frac{1}{2}q}{(n+1)^2} \right. \\ &\quad \left. + \frac{\frac{1}{12}p^4 - \frac{1}{3}p^2q + \frac{1}{6}q^2 - \frac{1}{6}p^3 + \frac{1}{2}pq + \frac{1}{12}p^2 - \frac{1}{6}q}{(n+1)^3} + \dots\right) \end{aligned} \quad (2.10)$$

as  $n \rightarrow \infty$ .

**Remark 2.1** Note that since  $W_n = 1/W_n(0, -\frac{1}{4})$ , it follows by setting  $(p, q) = (0, -\frac{1}{4})$  in (2.10) that

$$W_n \sim \frac{\pi}{2} \exp\left(-\frac{1}{4(n+1)} - \frac{1}{8(n+1)^2} - \frac{5}{96(n+1)^3} - \dots\right) \quad (2.11)$$

as  $n \rightarrow \infty$ .

The same procedure with the choice

$$(\lambda, \mu, \nu) = \left(\frac{1}{2}p, \frac{1}{4}p + \frac{1}{4}\Delta, \frac{1}{4}p + \frac{1}{4}\Delta\right)$$

in (2.2) and (2.6) leads to the following

**Theorem 2.2..** As  $n \rightarrow \infty$ , we have

$$R_n(p, q) \sim \frac{2^{-p}\pi e^{-p\gamma/2}}{\Gamma\left(\frac{1}{2} + \frac{1}{4}p + \frac{1}{4}\Delta\right) \Gamma\left(\frac{1}{2} + \frac{1}{4}p - \frac{1}{4}\Delta\right)} \exp\left(\sum_{j=1}^{\infty} \frac{b_j(p, q)}{(n + \frac{1}{2})^j}\right), \quad (2.12)$$

where the coefficients  $b_j(p, q)$  are given by

$$\begin{aligned} b_1(r, s) &= \frac{1}{8}p^2 - \frac{1}{4}q \quad \text{and} \\ b_j(r, s) &= \frac{pB_j}{2j} + \frac{(-1)^{j+1} \left( B_{j+1}(\frac{1}{4}p + \frac{1}{4}\Delta) + B_{j+1}(\frac{1}{4}p - \frac{1}{4}\Delta) - 2B_{j+1} \right)}{j(j+1)} \quad (j \geq 2). \end{aligned} \quad (2.13)$$

Thus we have the expansion

$$\begin{aligned}
R_n(p, q) \sim & \frac{2^{-p} \pi e^{-p\gamma/2}}{\Gamma\left(\frac{1}{2} + \frac{1}{4}p + \frac{1}{4}\Delta\right) \Gamma\left(\frac{1}{2} + \frac{1}{4}p - \frac{1}{4}\Delta\right)} \\
& \times \exp\left(\frac{\frac{1}{8}p^2 - \frac{1}{4}q}{n + \frac{1}{2}} + \frac{-\frac{1}{48}p^3 + \frac{1}{16}pq + \frac{1}{16}p^2 - \frac{1}{8}q}{(n + \frac{1}{2})^2}\right. \\
& \left. + \frac{\frac{1}{192}p^4 - \frac{1}{48}p^2q + \frac{1}{96}q^2 - \frac{1}{48}p^3 + \frac{1}{16}pq + \frac{1}{48}p^2 - \frac{1}{24}q}{(n + \frac{1}{2})^3} + \dots\right) \quad (2.14)
\end{aligned}$$

as  $n \rightarrow \infty$ .

The first two terms in the expansions (2.10) and (2.14) can be shown to agree with the expansions in inverse powers of  $n$  obtained in [10, Eqs. (4.3), (4.4)].

### 3 Asymptotic series expansions of the Wallis sequence

Some inequalities and asymptotic formulas associated with the Wallis sequence  $W_n$  can be found in [6, 13, 15, 17–21, 24–29, 31]. For example, Elezović *et al.* [15] showed that the following asymptotic expansion holds:

$$W_n \sim \frac{\pi}{2} \left( 1 - \frac{\frac{1}{4}}{n + \frac{5}{8}} + \frac{\frac{3}{256}}{(n + \frac{5}{8})^3} + \frac{\frac{3}{2048}}{(n + \frac{5}{8})^4} - \frac{\frac{51}{16384}}{(n + \frac{5}{8})^5} - \frac{\frac{75}{65536}}{(n + \frac{5}{8})^6} + \frac{\frac{2253}{1048576}}{(n + \frac{5}{8})^7} + \dots \right) \quad (3.1)$$

as  $n \rightarrow \infty$ . Deng *et al.* [13] proved that for all  $n \in \mathbb{N}$ ,

$$\frac{\pi}{2} \left( 1 - \frac{1}{4n + \alpha} \right) < W_n \leq \frac{\pi}{2} \left( 1 - \frac{1}{4n + \beta} \right) \quad (3.2)$$

with the best possible constants

$$\alpha = \frac{5}{2} \quad \text{and} \quad \beta = \frac{32 - 9\pi}{3\pi - 8} = 2.614909986\dots$$

In fact, Elezović *et al.* [15] have previously shown that  $\frac{5}{2}$  is the best possible constant for a lower bound of  $W_n$  of the type  $\frac{\pi}{2} \left( 1 - \frac{1}{4n + \alpha} \right)$ . Moreover, the authors pointed out that

$$W_n = \frac{\pi}{2} \left( 1 - \frac{1}{4n + \frac{5}{2}} \right) + O\left(\frac{1}{n^3}\right) \quad (n \rightarrow \infty).$$

Here, we will establish two more accurate asymptotic expansions for  $W_n$  (see Theorems 3.1 and 3.2) by making use of the fact that

$$W_n = \frac{\pi}{2} \cdot \frac{1}{n + \frac{1}{2}} \left[ \frac{\Gamma(n+1)}{\Gamma(n + \frac{1}{2})} \right]^2 = \frac{\pi}{2} \cdot \frac{\Gamma(n+1)^2}{\Gamma(n + \frac{1}{2})\Gamma(n + \frac{3}{2})}. \quad (3.3)$$

The following lemma is required in our present investigation.

**Lemma 3.1** (see [8]). *Let*

$$A(x) \sim \sum_{n=1}^{\infty} a_n x^{-n} \quad (x \rightarrow \infty)$$

*be a given asymptotical expansion. Then the composition  $\exp(A(x))$  has asymptotic expansion of the following form*

$$\exp(A(x)) \sim \sum_{n=0}^{\infty} b_n x^{-n} \quad (x \rightarrow \infty), \quad (3.4)$$

where

$$b_0 = 1, \quad b_n = \frac{1}{n} \sum_{k=1}^n k a_k b_{n-k} \quad (n \geq 1). \quad (3.5)$$

From (2.4), we find as  $n \rightarrow \infty$

$$W_n \sim \frac{\pi}{2} \exp \left( \sum_{j=1}^{\infty} \frac{\nu_j}{n^j} \right), \quad (3.6)$$

where the coefficients  $\nu_j$  are given by

$$\nu_j = \frac{(-1)^{j+1} \left( 2B_{j+1} - B_{j+1}(\frac{1}{2}) - B_{j+1}(\frac{3}{2}) \right)}{j(j+1)} \quad (j \geq 1). \quad (3.7)$$

Noting that (see [1, pp. 805–804])

$$B_n(1-x) = (-1)^n B_n(x), \quad (-1)^n B_n(-x) = B_n(x) + nx^{n-1} \quad (n \in \mathbb{N}_0)$$

and

$$B_n(\frac{1}{2}) = -(1 - 2^{1-n})B_n \quad (n \in \mathbb{N}_0),$$

we find that (3.7) can be written as

$$\nu_j = \frac{(-1)^{j+1} \left( (4 - 2^{1-j})B_{j+1} - (j+1) \cdot 2^{-j} \right)}{j(j+1)} \quad (j \geq 1). \quad (3.8)$$

Thus, we obtain the expansion

$$\begin{aligned} W_n \sim \frac{\pi}{2} \exp \left( -\frac{1}{4n} + \frac{1}{8n^2} - \frac{5}{96n^3} + \frac{1}{64n^4} - \frac{1}{320n^5} + \frac{1}{384n^6} - \frac{25}{7168n^7} \right. \\ \left. + \frac{1}{2048n^8} + \frac{29}{9216n^9} + \frac{1}{10240n^{10}} - \frac{695}{90112n^{11}} + \dots \right). \end{aligned} \quad (3.9)$$



By Lemma 3.1, we then obtain from (3.6)

$$W_n \sim \frac{\pi}{2} \sum_{j=0}^{\infty} \frac{\mu_j}{n^j}, \quad (3.10)$$

where the coefficients  $\mu_j$  are given by the recurrence relation

$$\mu_0 = 1, \quad \mu_j = \frac{1}{j} \sum_{k=1}^j k \nu_k \mu_{j-k} \quad (j \geq 1). \quad (3.11)$$

and the  $\nu_j$  are given in (3.8). This produces the expansion in inverse powers of  $n$  given by

$$\begin{aligned} W_n \sim \frac{\pi}{2} \left( 1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \frac{625}{65536n^6} - \frac{1843}{262144n^7} \right. \\ \left. + \frac{24323}{8388608n^8} + \frac{61477}{33554432n^9} - \frac{14165}{268435456n^{10}} - \frac{8084893}{1073741824n^{11}} + \dots \right) \end{aligned} \quad (3.12)$$

as  $n \rightarrow \infty$ .

**Theorem 3.1.** *The Wallis sequence has the following asymptotic expansion:*

$$W_n \sim \frac{\pi}{2} \left( 1 + \sum_{\ell=1}^{\infty} \frac{\alpha_{\ell}}{(n + \beta_{\ell})^{2\ell-1}} \right) \quad (n \rightarrow \infty), \quad (3.13)$$

where  $\alpha_{\ell}$  and  $\beta_{\ell}$  are given by the pair of recurrence relations

$$\alpha_{\ell} = \mu_{2\ell-1} - \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \quad (\ell \geq 2) \quad (3.14)$$

and

$$\beta_{\ell} = -\frac{1}{(2\ell-1)\alpha_{\ell}} \left\{ \mu_{2\ell} + \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \right\} \quad (\ell \geq 2), \quad (3.15)$$

with  $\alpha_1 = -\frac{1}{4}$  and  $\beta_1 = \frac{5}{8}$ . Here  $\mu_j$  are given by the recurrence relation (3.11).

*Proof.* Let

$$W_n \sim \frac{\pi}{2} \left( 1 + \sum_{\ell=1}^{\infty} \frac{\alpha_{\ell}}{(n + \beta_{\ell})^{2\ell-1}} \right) \quad (n \rightarrow \infty),$$

where  $\alpha_{\ell}$  and  $\beta_{\ell}$  are real numbers to be determined. This can be written as

$$\frac{2}{\pi} W_n \sim 1 + \sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \left( 1 + \frac{\beta_j}{n} \right)^{-2j+1}. \quad (3.16)$$

Direct computation yields

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \left(1 + \frac{\beta_j}{n}\right)^{-2j+1} &\sim \sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \sum_{k=0}^{\infty} \binom{-2j+1}{k} \frac{\beta_j^k}{n^k} \\
&\sim \sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \sum_{k=0}^{\infty} (-1)^k \binom{k+2j-2}{k} \frac{\beta_j^k}{n^k} \\
&\sim \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \alpha_{k+1} \beta_{k+1}^{j-k-1} (-1)^{j-k-1} \binom{j+k-1}{j-k-1} \frac{1}{n^{j+k}},
\end{aligned}$$

which can be written as

$$\sum_{j=1}^{\infty} \frac{\alpha_j}{n^{2j-1}} \left(1 + \frac{\beta_j}{n}\right)^{-2j+1} \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{n^j}. \quad (3.17)$$

It then follows from (3.16) and (3.17) that

$$\frac{2}{\pi} W_n \sim 1 + \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{n^j}. \quad (3.18)$$

On the other hand, we have from (3.10) that

$$\frac{2}{\pi} W_n \sim 1 + \sum_{j=1}^{\infty} \frac{\mu_j}{n^j}. \quad (3.19)$$

Equating coefficients of  $n^{-j}$  on the right-hand sides of (3.18) and (3.19), we obtain

$$\mu_j = \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \alpha_k \beta_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \quad (j \in \mathbb{N}). \quad (3.20)$$

Setting  $j = 2\ell - 1$  and  $j = 2\ell$  in (3.20), respectively, we find

$$\mu_{2\ell-1} = \sum_{k=1}^{\ell} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \quad (3.21)$$

and

$$\mu_{2\ell} = - \sum_{k=1}^{\ell} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}. \quad (3.22)$$

For  $\ell = 1$ , we obtain from (3.21) and (3.22)

$$\alpha_1 = \mu_1 = -\frac{1}{4} \quad \text{and} \quad \beta_1 = -\frac{\mu_2}{\alpha_1} = \frac{5}{8},$$

and for  $\ell \geq 2$  we have

$$\mu_{2\ell-1} = \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \alpha_\ell$$

and

$$\mu_{2\ell} = - \sum_{k=1}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - (2\ell-1) \alpha_\ell \beta_\ell.$$

We then obtain the recurrence relations (3.14) and (3.15). The proof of Theorem 3.1 is complete.  $\square$

We now give explicit numerical values of the first few  $\alpha_\ell$  and  $\beta_\ell$  by using the recurrence relations (3.14) and (3.15). This demonstrates the ease with which the constants  $\alpha_\ell$  and  $\beta_\ell$  in (3) can be determined. We find

$$\begin{aligned} \alpha_1 &= -\frac{1}{4}, \quad \beta_1 = \frac{5}{8}, \\ \alpha_2 &= \mu_3 - \alpha_1 \beta_1^2 = -\frac{11}{128} - \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^2 = \frac{3}{256}, \\ \beta_2 &= -\frac{\mu_4 + \alpha_1 \beta_1^3}{3\alpha_2} = -\frac{\frac{83}{2048} + \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^3}{3 \cdot \left(\frac{3}{256}\right)} = \frac{7}{12}, \\ \alpha_3 &= \mu_5 - \alpha_1 \beta_1^4 - 6\alpha_2 \beta_2^2 = -\frac{143}{8192} - \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^4 - 6 \cdot \left(\frac{3}{256}\right) \cdot \left(\frac{7}{12}\right)^2 = -\frac{53}{16384}, \\ \beta_3 &= -\frac{\mu_6 + \alpha_1 \beta_1^5 + 10\alpha_2 \beta_2^3}{5\alpha_3} = -\frac{\frac{625}{65536} + \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{8}\right)^5 + 10 \cdot \left(\frac{3}{256}\right) \cdot \left(\frac{7}{12}\right)^3}{5 \cdot \left(-\frac{53}{16384}\right)} = \frac{2113}{3816}. \end{aligned}$$

Continuation of this procedure then enables the following coefficients to be derived:

$$\begin{aligned} \alpha_4 &= \frac{224573}{93782016}, \quad \beta_4 = \frac{22119189899}{41134587264}, \\ \alpha_5 &= -\frac{596297240983745796931}{176651089583152098705408}, \quad \beta_5 = \frac{38909478384301921254232134966821}{73585322683584986068354328660352}. \end{aligned}$$

We then obtain the following explicit asymptotic expansion:

$$\begin{aligned} W_n \sim \frac{\pi}{2} \left( 1 - \frac{\frac{1}{4}}{n + \frac{5}{8}} + \frac{\frac{3}{256}}{\left(n + \frac{7}{12}\right)^3} - \frac{\frac{53}{16384}}{\left(n + \frac{2113}{3816}\right)^5} \right. \\ \left. + \frac{\frac{224573}{93782016}}{\left(n + \frac{22119189899}{41134587264}\right)^7} - \frac{\frac{596297240983745796931}{176651089583152098705408}}{\left(n + \frac{38909478384301921254232134966821}{73585322683584986068354328660352}\right)^9} + \dots \right). \end{aligned} \quad (3.23)$$

Thus, we would appear to obtain an alternating odd-type asymptotic expansion for  $W_n$ . From a computational viewpoint, (3.23) is an improvement on the formulas (3.12) and (3.1).

**Theorem 3.2.** *The Wallis sequence has the following asymptotic expansion:*

$$W_n \sim \frac{\pi}{2} \exp \left( \sum_{\ell=1}^{\infty} \frac{\omega_\ell}{\left(n + \frac{1}{2}\right)^{2\ell-1}} \right) \quad (n \rightarrow \infty) \quad (3.24)$$

with the coefficients  $\omega_\ell$  given by the recurrence relation

$$\omega_1 = -\frac{1}{4} \quad \text{and} \quad \omega_\ell = \nu_{2\ell-1} - \sum_{k=1}^{\ell-1} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \quad (\ell \geq 2), \quad (3.25)$$

where the  $\nu_j$  are given in (3.8).

*Proof.* Let

$$W_n \sim \frac{\pi}{2} \exp \left( \sum_{\ell=1}^{\infty} \frac{\omega_\ell}{(n + \frac{1}{2})^{2\ell-1}} \right) \quad (n \rightarrow \infty),$$

where  $\omega_\ell$  are real numbers to be determined. This can be written as

$$\ln \left( \frac{2}{\pi} W_n \right) \sim \sum_{j=1}^{\infty} \frac{\omega_j}{n^{2j-1}} \left( 1 + \frac{1}{2n} \right)^{-2j+1}.$$

The choice  $\beta_j = \frac{1}{2}$  in (3.17), with  $\alpha_j$  replaced by  $\omega_j$ , yields

$$\sum_{j=1}^{\infty} \frac{\omega_j}{n^{2j-1}} \left( 1 + \frac{1}{2n} \right)^{-2j+1} \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \omega_k \left( \frac{1}{2} \right)^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{n^j}.$$

We then obtain

$$\ln \left( \frac{2}{\pi} W_n \right) \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \omega_k \left( \frac{1}{2} \right)^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{n^j}. \quad (3.26)$$

On the other hand, we have from (3.6) that

$$\ln \left( \frac{2}{\pi} W_n \right) \sim \sum_{j=1}^{\infty} \frac{\nu_j}{n^j}. \quad (3.27)$$

Equating coefficients of  $n^{-j}$  on the right-hand sides of (3.26) and (3.27), we obtain

$$\nu_j = \sum_{k=1}^{\lfloor \frac{j+1}{2} \rfloor} \omega_k \left( \frac{1}{2} \right)^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \quad (j \in \mathbb{N}). \quad (3.28)$$

Setting  $j = 2\ell - 1$  in (3.28), we find

$$\nu_{2\ell-1} = \sum_{k=1}^{\ell} \omega_k \left( \frac{1}{2} \right)^{2\ell-2k} \binom{2\ell-2}{2\ell-2k}. \quad (3.29)$$

Substitution of  $\ell = 1$  in (3.29) yields  $\omega_1 = \nu_1 = -\frac{1}{4}$ , and for  $\ell \geq 2$  we have

$$\nu_{2\ell-1} = \sum_{k=1}^{\ell-1} \omega_k \left( \frac{1}{2} \right)^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \omega_\ell.$$

We then obtain the recurrence relation (3.25). The proof of Theorem 3.2 is complete.  $\square$

**Remark 3.1.** Setting  $j = 2\ell$  in (3.28), we find

$$\nu_{2\ell} = -\sum_{k=1}^{\ell} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}. \quad (3.30)$$

For  $\ell = 1$  in (3.30) this yields  $\omega_1 = -2\nu_2 = -\frac{1}{4}$ , and for  $\ell \geq 2$  we have

$$\nu_{2\ell} = -\sum_{k=1}^{\ell-1} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - \left(\ell - \frac{1}{2}\right) \omega_{\ell}.$$

We then obtain the alternative recurrence relation for the coefficients  $\omega_j$  in (3.24) in terms of the even coefficients  $\nu_j$ :

$$\omega_1 = -\frac{1}{4} \quad \text{and} \quad \omega_{\ell} = -\frac{2}{2\ell-1} \left\{ \nu_{2\ell} + \sum_{k=1}^{\ell-1} \omega_k \left(\frac{1}{2}\right)^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \right\} \quad (\ell \geq 2). \quad (3.31)$$

Hence, from (3.24), we obtain the following explicit asymptotic expansion:

$$W_n \sim \frac{\pi}{2} \exp \left( -\frac{\frac{1}{4}}{n + \frac{1}{2}} + \frac{\frac{1}{96}}{(n + \frac{1}{2})^3} - \frac{\frac{1}{320}}{(n + \frac{1}{2})^5} + \frac{\frac{17}{7168}}{(n + \frac{1}{2})^7} - \frac{\frac{31}{9216}}{(n + \frac{1}{2})^9} + \dots \right). \quad (3.32)$$

This would appear to be an alternating odd-type expansion for  $W_n$ . From a computational viewpoint, (3.32) is an improvement on the formulas (2.11) and (3.9).

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